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Time-Delay Induced Reentrance Phenomenon in a Triple-Well Potential System Driven by Cross-Correlated Noises

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Abstract The effects of the time delay on the stationary properties of a triple-well potential system driven by the cross-correlated multiplicative and additive noises are investigated. The stationary probability distribution function (SPDF) is obtained by means of a numerical simulation method. The results indicate that: (i) The time delay can induce the reentrance phenomenon with increasing the delay time; (ii) The cross-correlation between the multiplicative and additive noises induces the symmetry breaking of the SPDF.

Keywords Time delay · Cross-correlated noises · Reentrance phenomenon · Triple-well potential system

1 Introduction

Noise is immanent in any open systems involving up-take and dissipation of energy, while time delay usually arises due to finite transmission time of the signals or other key quantities (such as information, matter, energy, and so on) [3, 4, 7]. Therefore, noise and time delay are two important elements in many physical systems. In many cases, the systems with time delay and noise can be described in terms of the stochastic delay differential equation (SDDE) [7]. Recently, the delay Fokker-Planck equation corresponding to the SDDE has been derived. However, it just can be approximately solved for the case of small delay time [3, 4, 7]. Thus, the direct numerical simulation of the SDDE is usually used to investigate the stochastic systems with time delay, especially for the cases of large delay times [9, 16, 19].

In many natural and physical situations, the time delay is usually used to describe an intrinsic delay mechanism or an introduction of the time-delayed feedback, which implies that the dissipative evolution depends on the state of the system in a shifted previous time [1]. Thus, the effects of the time delay on the nonlinear stochastic systems have recently gained considerable attention [9, 16, 17]. It has been revealed that the interplay between the noise

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and time delay can generate very complex dynamical behaviors, such as the time-delay induced transition [5, 16], the time-delay induced critical behavior [9], and so on. On the other hand, since Fulinski et al. [6] pointed out that noises in some stochastic processes may have a common origin and thus can be cross-correlated, the effects of the cross-correlation between noise terms have been extensively studied in different systems [8, 12–15, 20]. Thus, the investigation of the effects of the interplay between the time delay and cross-correlated noises in different systems would be an interesting issue.

The reentrance phenomenon was firstly observed in an analysis of a chemical reaction system driven by one colored noise, which showed that the system undergoes a purely noise-induced transition from a monostable regime to a bistable one, then to the monostable regime as the noise correlation time increases [2]. This phenomenon has also been found in a bistable system subjected to the cross-correlated noises [10]. Motivated by the previous studies on the time-delay induced transition and the reentrance phenomenon, a question to be raised is if similar phenomenon can be induced by the delay time.

In this paper, based on a triple-well potential system driven by the cross-correlated noises [8], a nonlinear time-delayed feedback is introduced into this system. The effects of the time delay and noise correlation on the stationary properties of the system are investigated by means of a numerical simulation method. More interestingly, we will show that the time delay can induce the reentrance phenomenon in this system.

2 The Stationary Probability Distribution of the System

Recently, Ghosh et al. [8] considered an overdamped Brownian particle in a symmetric triple-well potential $U(x)$ kept in a thermal bath at temperature T and subjected to the cross-correlated multiplicative noise $\xi(t)$ and additive noise $\eta(t)$, in which the underlying idea rests on controlling the pathways of a parallel reaction by means of the cross-correlated noises. The middle well represents the reactant state, while the two side wells refer to the product states of the parallel reaction. The governing Langevin equation is given by Ghosh et al. [8]

$$\gamma \dot{x} = -U'(x) + x\xi(t) + \eta(t) + \Gamma(t), \tag{1}$$

where $U(x) = x^2(bx^2 - c)^2$ is the symmetric triple-well potential, b and c are the parameters of the potential, and γ is the dissipation constant. For simplicity, we assume in the present study that the noise terms have the following statistical properties

$$\langle \xi(t)\rangle = \langle \eta(t)\rangle = \langle \Gamma(t)\rangle = 0, \tag{2}$$

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{3}$$

$$\langle \eta(t)\eta(t')\rangle = 2\alpha\delta(t-t'), \tag{4}$$

$$\langle \Gamma(t)\Gamma(t')\rangle = 2Q\delta(t-t'), \tag{5}$$

and

$$\langle \xi(t)\eta(t')\rangle = \langle \xi(t')\eta(t)\rangle = 2\lambda\sqrt{D\alpha}\,\delta(t-t'). \tag{6}$$

D, α and Q denote the intensities of the multiplicative noise, additive noise and thermal fluctuation, respectively. Q is given by $Q = kT/\gamma$, where k is the Boltzmann constant and T is the temperature of the thermal bath. λ represents the strength of the cross-correlation between $\xi(t)$ and $\eta(t)$.

Here, we introduce a nonlinear time-delayed feedback into the system described by (1) with (2)–(6), and assume that the governing SDDE can be written as

$$\gamma \dot{x}(t) = 8bcx^3(t-\tau) - 6b^2x^5(t) - 2c^2x(t) + x(t)\xi(t) + \eta(t) + \Gamma(t), \tag{7}$$

where τ is the delay time. Equation (7) implies that the force acting on the Brownian particle depends not only on the present state of the system $x(t)$, but also on the state at earlier time $x(t-\tau)$. To obtain the stationary probability distribution function (SPDF) of the system for small and large delay times, we perform a series of numerical simulations, using the forward Euler algorithm [18]. The Box-Mueller algorithm is used to generate Gaussian white noise from two random numbers which are uniformly distributed in the unit interval [11].

According to the method proposed by Budini et al. [1], after making the following transformation

$$\eta_1(t) = \eta(t) - \lambda\sqrt{\frac{\alpha}{D}}\xi(t), \tag{8}$$

and substituting x_τ for $x(t-\tau)$, (7) can be equivalently rewritten as

$$\gamma \dot{x} = 8bcx_\tau^3 - 6b^2x^5 - 2c^2x + \left(x + \lambda\sqrt{\frac{\alpha}{D}}\right)\xi(t) + \eta_1(t) + \Gamma(t). \tag{9}$$

Thus $\eta_1(t)$ and $\xi(t)$ become independent, and have the following properties

$$\langle \eta_1(t) \rangle = 0, \tag{10}$$

$$\langle \eta_1(t)\eta_1(t') \rangle = 2\alpha(1-\lambda^2)\delta(t-t'), \tag{11}$$

and

$$\langle \eta_1(t)\xi(t') \rangle = \langle \eta_1(t')\xi(t) \rangle = 0. \tag{12}$$

Using the forward Euler algorithm with a small time step ($\Delta t = 0.001$) [18], (9) can be formally integrated as

$$\begin{aligned} x(t+\Delta t) - x(t) = {} & [8bcx^3(t-\tau) - 6b^2x^5(t) - 2c^2x(t)]\Delta t + [x(t) + \lambda\sqrt{\alpha/D}]N_1 \\ & + \frac{1}{2}[x(t) + \lambda\sqrt{\alpha/D}]N_1^2 + N_2 + N_3, \end{aligned} \tag{13}$$

where $N_1 = [-4D\Delta t \ln(\gamma_1)]^{1/2}\cos(2\pi\gamma_2)$, $N_2 = [-4\alpha(1-\lambda^2)\Delta t \ln(\gamma_3)]^{1/2}\cos(2\pi\gamma_4)$, $N_3 = [-4Q\Delta t \ln(\gamma_5)]^{1/2}\cos(2\pi\gamma_6)$, and γ_i $(i = 1, 2, \ldots, 6)$ is six independent random numbers distributed in the interval $(0, 1)$. The initial value of the numerical simulation is $x(0) = 0$ and $x(t) = 0$ as $t < \tau$. By a series of numerical simulations using (13), the SPDF of the system for various values of the noise parameter and delay time are obtained as follows.

Under the conditions of the strength of the cross-correlation between noise terms $\lambda = 0$ and small intensities of the multiplicative noise (D) and additive noise (α), the SPDF of the system for various values of the delay time is reported in Fig. 1. One can see from Fig. 1 that the time-delay induced transition can be observed. More interestingly, the reentrance phenomenon induced by the time delay occurs, i.e., with increasing the delay time τ, the SPDF of the system changes from a three peak structure to a two peak structure. Along with τ further increasing, the SPDF of the system goes back to a three peak structure, and

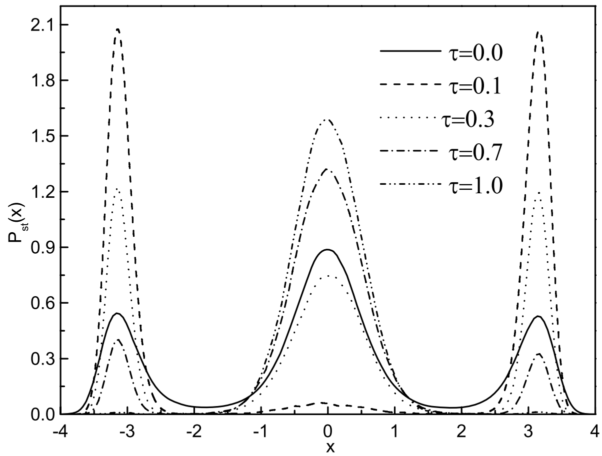

Fig. 1 The SPDF $P_{st}(x)$ vs. x for various values of the delay time. The parameters are $b = 0.1, c = 1.0, k = \gamma = 1.0$, $T = 0.44$, $D = \alpha = 0.01$ and $\lambda = 0$, which are the same as those used by Ghosh et al. [8]

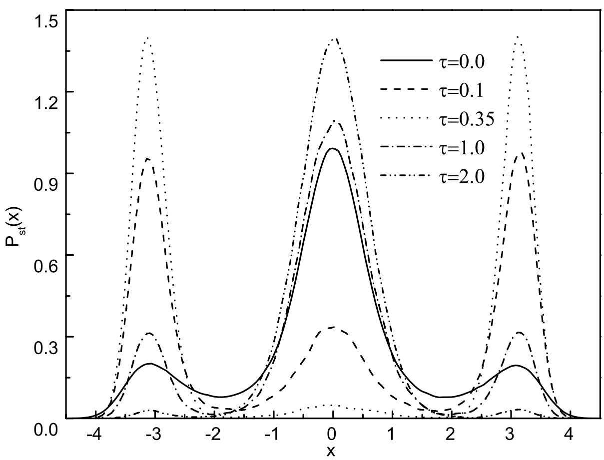

Fig. 2 The SPDF $P_{st}(x)$ vs. x with $D = \alpha = 0.1$ for various values of the delay time. The other parameters are the same as those in Fig. 1

ultimately becomes a single peak structure. When D and α take the larger values, the SPDF of the system is shown in Fig. 2 for the case of $\lambda = 0$. Figure 2 shows that with increasing τ, the same transition behavior induced by the delay time can also be observed. However, by comparing Fig. 1 with Fig. 2 one can clearly see that the critical value of the delay time, at which the delay-induced transition takes place, increases with D and α increasing. One should also note that for the case of $\lambda = 0$, the SPDF always remains symmetric, while the time delay changes the probability distribution in different well as τ increases.

For the case of $\lambda \neq 0$, the SPDF of the system for various values of the delay time is plotted in Figs. 3 and 4. From Figs. 3 and 4, one can clearly see that the time-delay induced reentrance phenomenon can also be observed for the case of $\lambda \neq 0$. Figures 3 and 4 show that the cross-correlation between the noise terms can not induce transition but it induces the symmetry breaking effect on the SPDF when D and α are small. That is, as λ increases, the left peak of the SPDF goes down and the right peak goes up for the case of $\tau \neq 0$. Moreover, with increasing τ, the symmetry breaking effect of the λ on the SPDF is suppressed by the time delay. However, when $\tau = 0$, as λ increases, the left peak of the SPDF goes up and the

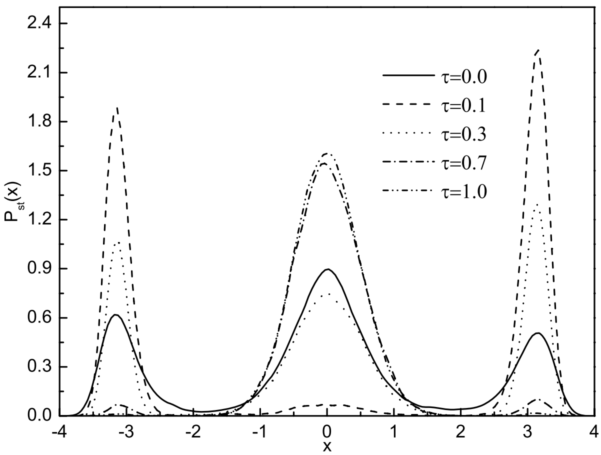

Fig. 3 The SPDF $P_{st}(x)$ vs. x with $\lambda = 0.5$ for various values of the delay time. The other parameters are the same as those in Fig. 1

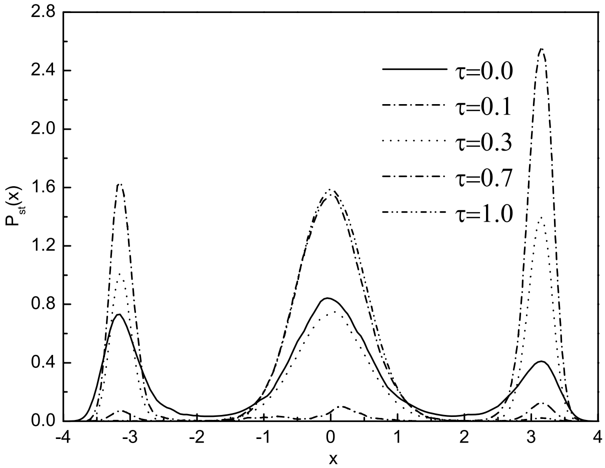

Fig. 4 The SPDF $P_{st}(x)$ vs. x with $\lambda = 1.0$ for various values of the delay time. The other parameters are the same as those in Fig. 1

right peak goes down. Thus, the λ plays an opposite role on the SPDF for the cases of $\tau = 0$ and $\tau \neq 0$.

3 Conclusion

We have numerically investigated the effects of the time delay and noise correlation on the stationary properties of a triple-well potential system driven by the cross-correlated multiplicative and additive white noises. It has been shown that the delay time can induce the reentrance phenomenon for the cases of both $\lambda = 0$ and $\lambda \neq 0$. Additionally, when $\lambda = 0$ and τ varies, the SPDF of the system always remains symmetric. However, for the case of $\lambda \neq 0$, a symmetry breaking effect on the SPDF can be observed by increasing λ and it can be suppressed with τ increasing. The λ plays an opposite role on the SPDF when $\tau = 0$ and $\tau \neq 0$. As D and α increase, the critical value of the delay time for the delay-induced transition increases accordingly. The results of this work suggest that the introduction of time-delayed feedback may serve as a technique for preferentially controlling a pathway for a parallel chemical reaction. This control becomes more flexible due to the combination of the time-delayed feedback with the noise correlation.

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